Squares, Cubes, and Beyond

Learning Objectives

- Simplify square roots.
- Find cube roots.
- Simplify expressions with odd and even roots.

Introduction

Radical expressions are expressions that contain radicals. Radical expressions come in many forms, from simple and familiar, such as $\sqrt{16}$, to quite complicated, as in $\sqrt[3]{250x^4y}$. In addition to square roots, there are radicals called cube roots, fourth roots, fifth roots, and so on. Using factoring, you can simplify these radical expressions, too.

Simplifying Square Roots

Radical expressions will sometimes include variables as well as numbers. Consider the expression $\sqrt{9x^6y^4}$. To simplify a radical expression such as this, you can use factoring, but you'll have to apply the rules of exponents, too. Let's try it.

Example	
Problem Simplify. $\sqrt{9x^6y^4}$	
$\sqrt{3 \cdot 3 \cdot x^6y^4}$	Factor the coefficient 9 into $3 \cdot 3$.
$\sqrt{3 \cdot 3 \cdot x^2 \cdot x^2 \cdot x^2 \cdot y^2 \cdot y^2}$	Factor variables into squares.

	$\sqrt{3^2} \cdot \sqrt{x^2} \cdot \sqrt{x^2} \cdot \sqrt{x^2} \cdot \sqrt{y^2} \cdot \sqrt{y^2}$	Write $3 \cdot 3$ as 3^2 and separate into individual radicals.
	$3 \cdot x \cdot x \cdot x \cdot y \cdot y$	Simplify, using the rule that $\sqrt{x^2} = x$.
	$3x^3y^2$	Rewrite the expression with constants in front and using exponents for the variables.
Answer	$\sqrt{9x^6y^4} = 3x^3y^2$	

The goal is to find factors under the radical that are perfect squares so that you can take their square root. Let's repeat the example above and focus on finding identical pairs of factors.

Problem	**Example** **Simplify.** $\sqrt{9x^6y^4}$	
	$\sqrt{3 \cdot 3 \cdot x^3 \cdot x^3 \cdot y^2 \cdot y^2}$	Factor to find identical pairs.
	$\sqrt{(3^2) \cdot (x^3)^2 \cdot (y^2)^2}$	Rewrite the pairs as perfect squares.
	$\sqrt{3^2} \cdot \sqrt{(x^3)^2} \cdot \sqrt{(y^2)^2}$	Separate into individual radicals.
	$3x^3y^2$	Simplify, using the rule that $\sqrt{x^2} = x$.
Answer	$\sqrt{9x^6y^4} = 3x^3y^2$	

Variable factors with even exponents can be written as squares. In the example above, $x^6 = x^3 \cdot x^3 = \left(x^3\right)^2$ and $y^4 = y^2 \cdot y^2 = \left(y^2\right)^2$. Let's try to simplify another radical expression.

Example		
Problem	**Simplify.** $\sqrt{49x^{10}y^8}$	
	$\sqrt{7 \cdot 7 \cdot x^5 \cdot x^5 \cdot y^4 \cdot y^4}$	Look for squared numbers and variables. Factor 49 into $7 \cdot 7$, x^{10} into $x^5 \cdot x^5$, and y^8 into $y^4 \cdot y^4$.
	$\sqrt{7^2 \cdot \left(x^5\right)^2 \cdot \left(y^4\right)^2}$	Rewrite the pairs as squares.
	$\sqrt{7^2} \cdot \sqrt{\left(x^5\right)^2} \cdot \sqrt{\left(y^4\right)^2}$	Separate the squared factors into individual radicals.
	$7 \cdot x^5 \cdot y^4$	Take the square root of each radical using the rule that $\sqrt{x^2} = x$.
	$7x^5y^4$	Multiply.
Answer	$\sqrt{49x^{10}y^8} = 7x^5y^4$	

You find that the square root of $49x^{10}y^8$ is $7x^5y^4$. In order to check this calculation, you could square $7x^5y^4$, hoping to arrive at $49x^{10}y^8$. And, in fact, you would get this expression if you evaluated $\left(7x^5y^4\right)^2$.

Absolute Value

Take a moment to think about two radical expressions: $\sqrt{900}$ and $\sqrt{a^3 b^5 c^7}$. You would use the same techniques to simplify either one of these: find squares within the radical, rewrite the expression as the product of separate radicals, simplify and multiply.

There is an added issue when you are taking the root of a radical expression that contains variables. Recall that the root of an integer, such as $\sqrt{900}$, is defined to be nonnegative. This means that although both 30^2 and $(-30)^2$ are equal to 900, $\sqrt{900}$ is defined *only* as 30. This is the idea behind 30 being the **principal root** of 900.

But it is not as straightforward with radical expressions that contain variables. Consider the expression $\sqrt{x^2}$. This looks like it should be equal to x, right? Let's test some values for x and see what happens.

In the chart below, look along each row and determine whether the value of x is the same as the value of $\sqrt{x^2}$. Where are they equal? Where are they not equal?

After doing that for each row, look again and determine whether the value of $\sqrt{x^2}$ is the same as the value of $|x|$.

x	x^2	$\sqrt{x^2}$	$\lvert x \rvert$
-5	25	5	5
-2	4	2	2
0	0	0	0
6	36	6	6
10	100	10	10

Notice in cases where x is a negative number, $\sqrt{x^2} \neq x$! (This happens because the process of squaring the number loses the negative sign, since a negative times a negative is a positive.) However, in all cases $\sqrt{x^2} = |x|$. You need to consider this fact when simplifying radicals that contain variables, because by definition $\sqrt{x^2}$ is always nonnegative.

Taking the Square Root of a Radical Expression
When finding the square root of an expression that contains variables raised to a power, consider that $\sqrt{x^2} = |x|$.
Examples: $\sqrt{9x^2} = 3\,|x|$, and $\sqrt{16x^2y^2} = 4\,|xy|$

Look at how this idea is applied in this next example.

	Example
Problem	**Simplify.** $\sqrt{a^3b^5c^2}$
$\sqrt{a^2 \cdot a \cdot b^4 \cdot b \cdot c^2}$	Factor to find variables with even exponents.
$\sqrt{a^2 \cdot a \cdot \left(b^2\right)^2 \cdot b \cdot c^2}$	Rewrite b^4 as $\left(b^2\right)^2$.
$\sqrt{a^2} \cdot \sqrt{\left(b^2\right)^2} \cdot \sqrt{c^2} \cdot \sqrt{a \cdot b}$	Separate the squared factors into individual radicals.
$|a| \cdot b^2 \cdot |c| \cdot \sqrt{a \cdot b}$	Take the square root of each radical. Remember that $\sqrt{a^2} = |a|$.
$\left|ab^2c\right|\sqrt{ab}$	Simplify and multiply. The entire quantity ab^2c can be enclosed in the absolute value sign because b^2 will be

positive, so its inclusion has no effect.

Answer $\sqrt{a^3b^5c^2} = |ab^2c|\sqrt{ab}$

Simplify. $\sqrt{72y^2z^3}$

A) $36yz\sqrt{z}$

B) $6\sqrt{2y^2z^3}$

C) $6yz\sqrt{2z}$

D) $6|yz|\sqrt{2z}$

+ Show/Hide Answer

Finding Cube Roots

While square roots are probably the most common radical, you can also find the third root, the fifth root, the 10^{th} root, or really any other n^{th} root of a number. Just as the square root is a number that, when squared, gives the radicand, the **cube root** is a number that, when cubed, gives the radicand. Cubing a number is the same as taking it to the third power: 2^3 is 2 cubed, so the cube root of 2^3 is 2.

The cube root of a number is written with a small number 3, called the **index**, just outside and above the radical symbol. It looks like $\sqrt[3]{\ }$. This little 3 distinguishes cube roots from square roots, which are written without a small number outside and above the radical symbol.

Be careful to distinguish between $\sqrt[3]{x}$, the cube root of x, and $3\sqrt{x}$, three *times* the *square* root of x. They may look similar at first, but they lead you to much different expressions!

Example		
Problem	**Simplify.** $\sqrt[3]{8}$	
	$2 \cdot 2 \cdot 2$	Ask yourself, "What number can I multiply by itself, and then by itself again, to get 8?"
Answer	$\sqrt[3]{8} = 2$	

Another approach to simplifying a cube root is to use factoring. Let's explore factoring with the expression $\sqrt[3]{125}$. You can read this as "the third root of 125" or "the cube root of 125." To simplify this expression, look for a number that, when multiplied by itself two times (for a total of three identical factors), equals 125. Let's factor 125 and find that number.

Example		
Problem	**Simplify.** $\sqrt[3]{125}$	
	$\sqrt[3]{5 \cdot 25}$	125 ends in 5, so you know that 5 is a factor. Expand 125 into $5 \cdot 25$.
	$\sqrt[3]{5 \cdot 5 \cdot 5}$	Factor 25 into 5 and 5.
	$\sqrt[3]{5^3}$	The factors are $5 \cdot 5 \cdot 5$, or 5^3.
Answer	$\sqrt[3]{125} = 5$	

The prime factors of 125 are $5 \cdot 5 \cdot 5$, which can be rewritten as 5^3. The cube root of a cubed number is the number itself, so

$\sqrt[3]{5^3} = 5$. You have found the cube root, the three identical factors that when multiplied together give 125. 125 is known as a **perfect cube** because its cube root is an integer.

Here's an example of how to simplify a radical that is not a perfect cube.

Example		
Problem	**Simplify.** $\sqrt[3]{32m^5}$	
	$\sqrt[3]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot m^5}$	Factor 32 into prime factors.
	$\sqrt[3]{2^3 \cdot 2 \cdot 2 \cdot m^5}$	Since you are looking for the cube root, you need to find factors that appear 3 times under the radical. Rewrite $2 \cdot 2 \cdot 2$ as 2^3.
	$\sqrt[3]{2^3 \cdot 2 \cdot 2 \cdot m^3 \cdot m^2}$	Rewrite m^5 as $m^3 \cdot m^2$.
	$\sqrt[3]{2^3} \cdot \sqrt[3]{2 \cdot 2} \cdot \sqrt[3]{m^3} \cdot \sqrt[3]{m^2}$	Rewrite the expression as a product of multiple radicals.
	$2 \cdot \sqrt[3]{4} \cdot m \cdot \sqrt[3]{m^2}$	Simplify and multiply.
Answer	$\sqrt[3]{32m^5} = 2m\sqrt[3]{4m^2}$	

Simplify. $\sqrt[3]{64h^6}$

A) $8h^3$

B) $8\sqrt{h^6}$

C) $4 + h^2$

D) $4h^2$

+ Show/Hide Answer

Simplifying Expressions with Odd and Even Roots

There is one interesting fact about cube roots that is not true of square roots. Negative numbers can't have real number square roots, but negative numbers can have real number cube roots! What is the cube root of -8? $\sqrt[3]{-8} = -2$ because $-2 \cdot -2 \cdot -2 = -8$. Remember, when you are multiplying an odd number of negative numbers, the result is negative! In the example below, notice how $\sqrt[3]{(-1)^3} = -1$ is used to simplify the radical.

Example		
Problem	**Simplify.** $\sqrt[3]{-27x^4y^3}$	
	$\sqrt[3]{-1 \cdot 27 \cdot x^4 \cdot y^3}$ $\sqrt[3]{(-1)^3 \cdot (3)^3 \cdot x^3 \cdot x \cdot y^3}$	Factor the expression into cubes.
	$\sqrt[3]{-1^3} \cdot \sqrt[3]{(3)^3} \cdot \sqrt[3]{x^3} \cdot \sqrt[3]{x} \cdot \sqrt[3]{y^3}$	Separate the cubed factors into individual radicals.
	$-1 \cdot 3 \cdot x \cdot y \cdot \sqrt[3]{x}$	Simplify the cube roots.
Answer	$\sqrt[3]{-27x^4y^3} = -3xy\sqrt[3]{x}$	

You could check your answer by performing the inverse operation. If you are right, when you cube $-3xy\sqrt[3]{x}$ you should get $-27x^4y^3$.

$$\left(-3xy\sqrt[3]{x}\right)\left(-3xy\sqrt[3]{x}\right)\left(-3xy\sqrt[3]{x}\right)$$

$$-3\cdot -3\cdot -3\cdot x\cdot x\cdot x\cdot y\cdot y\cdot y\cdot \sqrt[3]{x}\cdot \sqrt[3]{x}\cdot \sqrt[3]{x}$$

$$-27\cdot x^3\cdot y^3\cdot \sqrt[3]{x^3}$$

$$-27x^3y^3\cdot x$$

$$-27x^4y^3$$

So, you can find the odd root of a negative number, but you cannot find the even root of a negative number. This means you can simplify the radicals $\sqrt[3]{-81}$, $\sqrt[5]{-64}$, and $\sqrt[7]{-2187}$, but you cannot simplify the radicals $\sqrt{-100}$, $\sqrt[4]{-16}$, or $\sqrt[6]{-2,500}$.

Let's look at another example.

Problem	**Example** **Simplify.** $\sqrt[3]{-24a^5}$	
	$\sqrt[3]{-1\cdot 8\cdot 3\cdot a^5}$	Factor -24 to find perfect cubes. Here, -1 and 8 are the perfect cubes.
	$\sqrt[3]{(-1)^3\cdot 2^3\cdot 3\cdot a^3\cdot a^2}$	Factor variables. You are looking for cube exponents, so you factor a^5 into a^3 and a^2.
	$\sqrt[3]{-1^3}\cdot \sqrt[3]{2^3}\cdot \sqrt[3]{a^3}\cdot \sqrt[3]{3\cdot a^2}$	Separate the factors into individual radicals.
	$-1\cdot 2\cdot a\cdot \sqrt[3]{3\cdot a^2}$	Simplify, using the property $\sqrt[3]{x^3}=x$.
	$-2a\sqrt[3]{3a^2}$	This is the simplest form of this expression; all

		cubes have been pulled out of the radical expression.
Answer	$\sqrt[3]{-24a^5} = -2a\sqrt[3]{3a^2}$	

The steps to consider when simplifying a radical are outlined below.

Simplifying a radical

When working with exponents and radicals:

- If n is odd, $\sqrt[n]{x^n} = x$.
- If n is even, $\sqrt[n]{x^n} = |x|$. (The absolute value accounts for the fact that if x is negative and raised to an even power, that number will be positive, as will the n th principal root of that number.)

Example

Problem	**Simplify.** $\sqrt{100x^2y^4}$	
	$\sqrt{10 \cdot 10 \cdot x^2 \cdot y^4}$	Separate factors; look for squared numbers and variables. Factor 100 into $10 \cdot 10$.
	$\sqrt{10 \cdot 10 \cdot x^2 \cdot \left(y^2\right)^2}$	Factor y^4 into $\left(y^2\right)^2$.
	$\sqrt{10^2} \cdot \sqrt{x^2} \cdot \sqrt{\left(y^2\right)^2}$	Separate the squared factors into individual radicals.
	$10 \cdot \lvert x \rvert \cdot y^2$	Take the square root of each radical. Since you do not know whether x is

		positive or negative, use $\|x\|$ to account for both possibilities, thereby guaranteeing that your answer will be positive.
	$10\|x\|y^2$	Simplify and multiply.
Answer	$\sqrt{100x^2y^4} = 10\|x\|y^2$	

You can check your answer by squaring it to be sure it equals $100x^2y^4$.

Summary

A radical expression is a mathematical way of representing the n th root of a number. Square roots and cube roots are the most common radicals, but a root can be any number. To simplify radical expressions, look for exponential factors within the radical, and then use the property $\sqrt[n]{x^n} = x$ if n is odd, and $\sqrt[n]{x^n} = |x|$ if n is even to pull out quantities. All rules of integer operations and exponents apply when simplifying radical expressions.

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