Solving Exponential and Logarithmic Equations

Learning Objectives

- Solve exponential equations.
- Solve logarithmic equations.

Introduction

As you know, algebra often requires you to solve equations to find unknown values. This is also true for exponential and logarithmic equations. There are some strategies that you can use, along with some properties you've learned, that you can use to solve those equations.

Solving Exponential Equations

You may be able to look at an equation like $4^x = 16$ and solve it by asking yourself, "4 to what power is 16? 4^2 is 16, so $x = 2$." Equations like $4^x = 17$ are a bit more difficult. You know x must be a little more than 2, because 17 is just a little more than 16. One way to find x with more precision, though, is by using **logarithms**.

When you have solved other algebraic equations, you often relied on the idea that you can change both sides of the equation in the same way and still get a true equation. This is true with logarithms, too: If $x = y$, then $\log_b x = \log_b y$, no matter what b is.

Let's look at this with an equation whose solution you already know: $4^x = 16$. You can use either the **common log** or the **natural log**. In the following example, you will use the common log.

Example		
Problem	Solve $4^x = 16$.	
	$4^x = 16$ $\log 4^x = \log 16$	Take the common logarithm of both sides. (Remember,

		when no base is written, that means the base is 10.) What can you do with that new equation?
	$\log 4^x = \log 16$ $x \log 4 = \log 16$	Use the power property of logarithms to simplify the logarithm on the left side of the equation.
	$x \log 4 = \log 16$ $x = \frac{\log 16}{\log 4}$	Remember that $\log 4$ is a number. You can divide both sides of the equation by $\log 4$ to get x by itself.
Answer	$x = \frac{1.204...}{0.602...} = 2$	Use a calculator to evaluate the logarithms and the quotient.

Just as you knew, $x = 2$. Now let's try it with our more difficult example, $4^x = 17$. The procedure is exactly the same.

Example		
Problem	Solve $4^x = 17$.	
	$4^x = 17$ $\log 4^x = \log 17$	Take the common logarithm of both sides.
	$\log 4^x = \log 17$ $x \log 4 = \log 17$	Use the power property of logarithms to simplify the logarithm on the left.
	$x \log 4 = \log 17$ $x = \frac{\log 17}{\log 4}$	Divide both sides by $\log 4$ to get x by itself.
Answer	$x = \frac{1.230...}{0.602...} = 2.043...$	Use a calculator to evaluate the logarithms and the quotient.

You could have used either the common log or the natural log with the example above. You use one of these two bases, as you can then use your calculator to find the values.

Example

Problem	Solve $e^{2x} = 54$.	
	$e^{2x} = 54$ $\ln e^{2x} = \ln 54$	Since the base is e, use the natural logarithm. (If the base were 10, using common logarithms would be better.)
	$\ln e^{2x} = \ln 54$ $2x = \ln 54$	Remember that logarithms and exponential functions are inverses. When you have $\log_b b^m$, the logarithm undoes the exponent, and the result is just m. So $\ln e^{2x} = \log_e e^{2x} = 2x$.
	$x = \frac{\ln 54}{2}$	Divide both sides by 2 to get x by itself.
Answer	$x = 1.99449...$	Use a calculator to evaluate the logarithm and quotient on the right and you're done!

Another kind of exponential equation has exponential expressions on both sides. When the bases are the same, or the exponents are the same, you can just compare the parts that are different. Look at these examples.

Example

Problem	Solve $3^{2x+5} = 3^{3x-2}$.	
	$3^{2x+5} = 3^{3x-2}$	Here are two exponential expressions with the same base. If the two expressions are equal, then their exponents must be equal. (Think about that: if you have 3^a and 3^b, and $a \neq b$, then 3^a can't have the same value as 3^b.)

	$2x + 5 = 3x - 2$	So, write a new equation that sets the exponents equal to each other.
	$5 = x - 2$ $7 = x$	Solve the linear equation as you normally would.
Check	$3^{2(7)+5} = 3^{3(7)-2}$ $3^{19} = 3^{19}$	Test the solution in the original equation. No need to find 3^{19}. When both sides say the same thing, you know it's correct!
Answer	$x = 7$	

Example

Problem	Solve $(x+4)^8 = 7^8$.	
	$(x+4)^8 = 7^8$	Again, you have two exponential expressions that are equal to each other. In this case, both sides have the same exponent, and this means the *bases* must be equal.
	$x + 4 = 7$	Write a new equation that sets the bases equal to each other.
	$x = 3$	Solve the linear equation as you normally would.
Check	$(3+4)^8 = 7^8$ $7^8 = 7^8$	Test the solution in the original equation. No need to find 7^8. When both sides say the same thing, you know it's correct!
Answer	$x = 3$	

Solve $10^{3x-2} = 13$.

A) $x = 5$

B) $x = 1.03798...$

C) $x = 1.52164...$

D) $x = 3.11394...$

+ Show/Hide Answer

Solving Logarithmic Equations

There are several strategies you can use to solve logarithmic equations. The first is one you have used before: Rewrite the logarithmic equation as an exponential equation!

Example		
Problem	Solve $\ln x = 4.657$. Give x to the thousandths place.	
	$\ln x = 4.657$ $\log_e x = 4.657$ $e^{4.657} = x$	Remember that natural logarithms have a base of e. Rewrite this logarithm as an exponential equation.
Answer	$105.3196... = x$ $x \approx 105.320$	Use a calculator to evaluate $e^{4.657}$, and round to the nearest thousandth.

This works regardless of the base.

Example		
Problem	Solve $\log_7 x = 3.843$. Give x to the thousandths place.	
	$\log_7 x = 3.843$ $7^{3.843} = x$	Rewrite this logarithm as an exponential equation.
Answer	$1768.9345... = x$ $x \approx 1768.935$	Use a calculator to evaluate $7^{3.843}$ and round to the nearest thousandth.

Logarithmic equations may also involve inputs where the variable has a coefficient other than 1, or where the variable itself is squared. In these cases, you need to complete a few more steps in solving for the variable.

Example		
Problem	Solve $\log_5 3x^2 = 1.96$. Give x to the hundredths place.	
	$5^{1.96} = 3x^2$	Rewrite this logarithmic equation as an exponential equation.
	$23.44127... = 3x^2$	Evaluate $5^{1.96}$.
	$7.81375... = x^2$ $x = \pm 2.7953...$ $x \approx \pm 2.80$	Solve as you normally would. In this case, divide both sides by 3, then use the square root property to find the possible values for x. Don't forget that when using the square root

		property, both positive and negative roots must be considered. Round to the nearest hundredth.
Check	$\log_5 3x^2 = 1.96$ $\log_5 3(-2.80)^2 = 1.96$ $\log_5 3\,(7.84) = 1.96$ $\log_5 23.52 = 1.96$	Check your answer by substituting the value of x into the original equation. Because $(-2.80)^2$ and $(+2.80)^2$ are both positive, we don't need to check $+\,2.80$ separately.
	$\frac{\log 23.52}{\log 5} = 1.96$ $\frac{1.3714...}{0.6989...} = 1.96$ $1.96 = 1.96$	Apply the change of base formula to switch from base 5 to base 10. The check shows that with rounding accounted for, a true statement results, so you know that the answer is correct.
Answer	$x \approx \pm\, 2.80$	

The equations may also include more than one logarithm. You can use the properties of logarithms to combine these logarithms into one logarithm. **Note:** You'll find it helpful to record which properties you use at each step, both to help you be sure you're using them properly and as a way to help you find errors.

Example

Problem	Solve $2\log 3 + \frac{1}{2}\log 16 - \log 3 = \log x$.
$2\log 3 + \frac{1}{2}\log 16 - \log 3 = \log x$	First notice that all of the logarithms have the same base. (These are common logarithms, so the bases are all 10.) When using the properties, it is *absolutely necessary* that the bases are the same.
$\log 3^2 + \log 16^{\frac{1}{2}} - \log 3 = \log x$	Use the power property to rewrite $2\log 3$ as $\log 3^2$ and $\frac{1}{2}\log 16$ as $\log 16^{\frac{1}{2}}$.
$\log 9 + \log 4 - \log 3 = \log x$	Evaluate the exponents.
$\log(9 \cdot 4) - \log 3 = \log x$ $\log 36 - \log 3 = \log x$	Use the product property $\log_b MN = \log_b M + \log_b N$ to combine $\log 9 + \log 4$.
$\log\left(\frac{36}{3}\right) = \log x$ $\log 12 = \log x$	Use the quotient property $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$ to combine $\log 36 - \log 3$.
Answer $x = 12$	Since the logarithm of 12 and the logarithm of x are equal, x must equal 12.

Solve $\log x + \log 3 = \log 24$.

A) 0.460...

B) 2.892...

C) 8

D) 21

+ Show/Hide Answer

Summary

There are several strategies that can be used to solve equations involving exponents and logarithms. Taking logarithms of both sides is helpful with exponential equations. Rewriting a logarithmic equation as an exponential equation is a useful strategy. Using properties of logarithms is helpful to combine many logarithms into a single one.

Continue